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## LETTER TO THE EDITOR

### Rooted spiral trees in dimensions 2, 3 and 4

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**Abstract.** We have studied rooted spiral trees in dimensions 2, 3 and 4 by exact enumeration and by Monte Carlo methods. Numerical estimates of critical exponents thus obtained support the hypothesis that a dimensional reduction by four occurs in this problem. A possible mechanism of the origin of this dimensional reduction is proposed.

Lattice animals have been much studied in literature as lattice models of branched polymers in dilute solution, and they also describe the statistics of large percolation clusters below percolation threshold (Lubensky and Isaacson 1979, Stauffer 1979, 1981). For a directed variant of the lattice animal problem, some interesting results have been obtained in the last few years (Dhar 1986 and references therein). In particular, it was found that the directed animal problem shows a dimensional reduction by 1, while the original (undirected) animal problem shows a dimensional reduction by 2 (Parisi and Sourlas 1981, Stanley *et al* 1982, Cardy 1982, Family 1982). Recently, a different variant of this problem, incorporating a spiral constraint, has been proposed (Li and Zhou 1985, Bose and Ray 1987). Li and Zhou (1985) estimated the exponents for spiral bond trees in two dimensions using exact enumeration of all trees having at most 14 bonds, and showed that the exponents are quite different from the case with no spiral constraint.

In this letter, we describe the results of an exact enumeration and Monte Carlo study of the site version of this problem in dimensions 2, 3 and 4. For the two-dimensional case, we have determined the exact number of spiral trees and their moments of inertia for all trees having up to 22 sites on the square lattice. In addition, the number of trees was determined for three more sizes, up to 25 sites. These data were supplemented by a Monte Carlo study of spiral trees having up to 50 sites. In three and four dimensions, we generated all spiral trees having up to 12 and 9 sites respectively. Monte Carlo studies for these dimensions have not yet been performed. From the estimated critical exponents in these cases, we find numerical evidence for dimensional reduction by four in this problem. A possible mechanism to explain this dimensional reduction is proposed.

On any lattice, a set of sites mutually connected by nearest-neighbour bonds is called a tree if it contains no loops. If one of the sites is specified as the root, it is called a rooted tree. A rooted tree is said to have the spiral property if, for all sites belonging to the tree, the projection of the path connecting to the root onto a specified plane (say the  $xy$  plane) contains no left turns. Figure 1 shows a spiral tree on the square lattice.

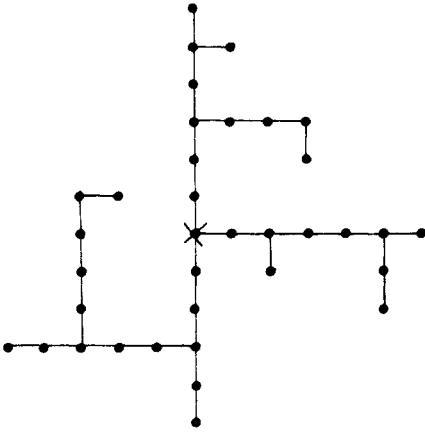


Figure 1. A rooted spiral tree on the square lattice. The root is denoted by a cross.

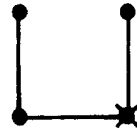


Figure 2. An example of a three-bond spiral tree that is not a four-site spiral tree. The root is denoted by a cross.

Other definitions of the spiral constraint are possible. In dimensions higher than 2, one may insist that the projections of the connecting paths onto more than one specified planes be spiralling. These additional constraints are interesting theoretically, as they presumably lead to dimensional reduction by more than four, but we shall not study these here.

It is useful to distinguish our definition from two other definitions that have been used in literature. Bose *et al* (1987) have studied unrooted spiral trees, where the root is not specified, and two trees differing only in the choice of root are not counted as distinct. The number of unrooted distinct spiral trees of a particular size is thus somewhat less than the rooted trees, though they are expected to be in the same universality class. Li and Zhou (1985) studied rooted bond spiral trees. These also contain configurations, such as are shown in figure 2, which are not allowed in the site version studied here.

Let  $A_n$  be the number of distinct rooted spiral site trees. The average values of the ( $xx$ ) and ( $zz$ ) components of the moment of inertia tensor over all trees of size  $n$  will be denoted by  $I_{n,pl}$  and  $I_{n,zz}$  (here  $z$  is any one of the directions perpendicular to the  $xy$  plane). These measure the average extension of the  $n$ -site spiral tree in the spiral plane directions and perpendicular to the spiral plane respectively. For large  $n$ ,  $A_n$  is expected to vary as  $\lambda^n n^{-\theta}$ , where  $\lambda$  is a constant and  $\theta$  is a critical exponent.  $I_{n,pl}$  and  $I_{n,zz}$  are expected to vary as powers of  $n$ :  $I_{n,pl} \sim n^{2\nu_{pl}+1}$  and  $I_{n,zz} \sim n^{2\nu_{\perp}+1}$ . These define the exponents  $\nu_{pl}$  and  $\nu_{\perp}$ . If  $d = 2$ ,  $\nu_{\perp}$  is not defined.

For the exact enumeration of spiral trees, we used the well known Martin algorithm (Martin 1974). The results for tree numbers and their average moments of inertia are shown in tables 1-3.

For analysing the exact data on square lattice tree numbers, we tried a sequential fit of the data to the form

$$\log A_n = n \log \lambda + B - \theta \log(n + \delta)$$

for three successive values of  $n$ , keeping  $\delta$  fixed (but adjustable). For  $\delta$  lying between 1.41 and 1.43, we found a satisfactory convergence of the successive estimates of  $\lambda$  and  $\theta$  to the values

$$\begin{aligned} \lambda &\approx 2.1166 \pm 0.0010 \\ \theta &\approx -1.307 \pm 0.006 \end{aligned} \quad (\text{square}).$$

**Table 1.** Exact data on square lattice spiral trees.

| $n$ | $A_n$   | $2\langle I_{n,pl} \rangle$ | $n$ | $A_n$         | $2\langle I_{n,pl} \rangle$ |
|-----|---------|-----------------------------|-----|---------------|-----------------------------|
| 1   | 1       | 0.000 000                   | 14  | 286 376       | 154.926 432                 |
| 2   | 4       | 1.000 000                   | 15  | 658 100       | 182.624 835                 |
| 3   | 14      | 3.142 857                   | 16  | 1 504 900     | 212.938 547                 |
| 4   | 40      | 6.800 000                   | 17  | 3 426 464     | 245.919 131                 |
| 5   | 105     | 12.266 667                  | 18  | 7 771 444     | 281.619 675                 |
| 6   | 268     | 19.656 716                  | 19  | 17 565 064    | 320.089 299                 |
| 7   | 674     | 28.919 881                  | 20  | 39 576 360    | 361.374 917                 |
| 8   | 1 660   | 40.159 036                  | 21  | 88 916 877    | 405.522 760                 |
| 9   | 4 021   | 53.513 056                  | 22  | 199 252 252   |                             |
| 10  | 9 612   | 69.074 906                  | 23  | 445 438 310   |                             |
| 11  | 22 734  | 86.926 014                  | 24  | 993 616 344   |                             |
| 12  | 53 276  | 107.140 851                 | 25  | 2 211 923 712 |                             |
| 13  | 123 916 | 129.787 372                 |     |               |                             |

**Table 2.** Exact data on spiral trees on a simple cubic lattice.

| $n$ | $A_n$       | $A_n\langle I_{n,pl} \rangle$ | $A_n\langle I_{n,zz} \rangle$ |
|-----|-------------|-------------------------------|-------------------------------|
| 1   | 1           | 0                             | 0                             |
| 2   | 6           | 2                             | 2                             |
| 3   | 41          | 38                            | 44                            |
| 4   | 260         | 472                           | 592                           |
| 5   | 1 568       | 4 722                         | 6 244                         |
| 6   | 9 190       | 41 652                        | 56 970                        |
| 7   | 53 090      | 338 702                       | 473 556                       |
| 8   | 303 900     | 2 597 720                     | 3 689 496                     |
| 9   | 1 727 691   | 19 056 884                    | 27 402 196                    |
| 10  | 9 767 426   | 134 997 622                   | 196 150 454                   |
| 11  | 54 966 550  | 929 820 026                   | 1 363 561 612                 |
| 12  | 308 138 528 | 6 258 509 116                 | 9 255 729 344                 |

**Table 3.** Exact data on  $d = 4$  hypercubic lattice spiral trees.

| $n$ | $A_n$      | $A_n\langle I_{n,pl} \rangle$ | $A_n\langle I_{n,zz} \rangle$ |
|-----|------------|-------------------------------|-------------------------------|
| 1   | 1          | 0                             | 0                             |
| 2   | 8          | 2                             | 2                             |
| 3   | 80         | 54                            | 60                            |
| 4   | 801        | 1 022                         | 1 212                         |
| 5   | 7 946      | 16 319                        | 20 202                        |
| 6   | 78 394     | 236 110                       | 300 648                       |
| 7   | 772 059    | 3 205 560                     | 4 157 398                     |
| 8   | 7 601 233  | 41 618 157                    | 54 621 874                    |
| 9   | 74 866 351 | 522 730 488                   | 691 254 410                   |

For the radius of gyration data, we used a sequential fit to the form

$$\log(I_n) = (2\nu_{pl} + 1) \log(n + \delta) + B + C/(n + \delta)^2.$$

For  $\delta$  lying between  $-0.50$  and  $-0.56$ , we found that the sequential fit values of  $\nu_{pl}$  are clustered around

$$2\nu_{pl} \approx 1.306 \pm 0.010 \quad (\text{square}).$$

It may be noted that these values differ significantly from the values deduced by Li and Zhou (1985) ( $\lambda = 2.662 \pm 0.006$ ,  $\theta = -1.19 \pm 0.03$ ,  $\nu = 0.577 \pm 0.01$ ) for spiral *bond* trees from the analysis of a shorter series. We have added three more terms to their series and reanalysed the data. The bond-tree numbers for  $n = 13-17$  on the square lattice are 3257 168, 9392 774, 26 946 680, 76 949 416, 218 826 412. This corrects the last two entries of Li and Zhou. Using the same fitting form as in the case of site trees, we obtain  $\lambda = 2.646 \pm 0.002$ ,  $\theta = -1.33 \pm 0.02$  for  $\delta$  near 1.9. Similar analysis for  $\nu_{pl}$  gives a value  $0.665 \pm 0.015$ . These values are in fair agreement with the corresponding values for site trees suggesting that these two are in the same universality class.

For the three-dimensional problem, similar sequential fits give

$$\begin{aligned} \lambda &= 5.368 \pm 0.003 \\ \theta &= -0.53 \pm 0.01 \\ \nu_{pl} &= 0.44 \pm 0.01 \\ \nu_{\perp} &= 0.52 \pm 0.01 \end{aligned} \quad (\text{simple cubic}).$$

For the four-dimensional case, our series is too short for a sophisticated analysis, but the data are quite consistent with

$$\begin{aligned} \lambda &\approx 9.85 \pm 0.03 & \theta &\approx 0 & (d = 4 \text{ hypercubic}) \\ \nu_{pl} &\approx \nu_{\perp} \approx 0.50 \pm 0.05. \end{aligned}$$

For the square lattice case, even though the series we generated is fairly long, the subjective error estimates given are quite large because of poor convergence of the estimates in the three to four similar extrapolation forms we tried. We have extended the numerical data for large  $n$  ( $n \leq 50$ ) by using the Monte Carlo incomplete enumeration methods (Dhar and Lam 1986). We determined the exact number of descendents of each tree generated, and also the radius of gyration. The average over the sample generated of the mean number of descendents and the average moment of inertia of a tree of  $n$  sites gives us a direct estimate of  $A_{n+1}/A_n$  and  $\langle I_{n,pl} \rangle$ .

A significant reduction in the variance of the Monte Carlo data is achieved if, instead of choosing survival probabilities in each generation so that the mean number of trees of  $n$  sites generated in one run is approximately one and making many runs (Dhar and Lam 1986), we do an exact enumeration up to size  $n_{\text{exact}} < N$  and incomplete enumeration for larger  $n$  so that each surviving tree with  $n \geq n_{\text{exact}}$  gives rise to approximately one surviving direct descendent on average (stratified sampling without replacement).

The standard error of the estimate is calculated from the spread in observed averages in different runs. In our Monte Carlo data, we made six different runs each for  $n_{\text{exact}} = 15$  and 16. This corresponds to a sample size of approximately  $2.5 \times 10^7$  trees for all  $n$  lying between 16 and 50. The data are summarised in table 4. The average number

Table 4. Monte Carlo data for spiral trees on the square lattice.

| $n$ | Mean moment of inertia | Mean number of descendents |
|-----|------------------------|----------------------------|
| 22  | 452.74 ± 0.46          | 2.2404 ± 0.0016            |
| 23  | 502.83 ± 0.47          | 2.2350 ± 0.0021            |
| 24  | 555.83 ± 0.63          | 2.2299 ± 0.0019            |
| 25  | 611.94 ± 0.84          | 2.2254 ± 0.0019            |
| 26  | 671.2 ± 1.2            | 2.2217 ± 0.0018            |
| 27  | 733.5 ± 1.3            | 2.2179 ± 0.0017            |
| 28  | 798.9 ± 1.7            | 2.2141 ± 0.0017            |
| 29  | 867.2 ± 2.3            | 2.2109 ± 0.0013            |
| 30  | 938.8 ± 2.8            | 2.2077 ± 0.0018            |
| 31  | 1013.6 ± 3.7           | 2.2048 ± 0.0022            |
| 32  | 1092.0 ± 4.8           | 2.2020 ± 0.0028            |
| 33  | 1173.4 ± 5.9           | 2.2995 ± 0.0026            |
| 34  | 1258.3 ± 7.0           | 2.1969 ± 0.0030            |
| 35  | 1346.7 ± 7.9           | 2.1950 ± 0.0032            |
| 36  | 1438.3 ± 9.4           | 2.1930 ± 0.0040            |
| 37  | 1533.4 ± 10.3          | 2.1905 ± 0.0037            |
| 38  | 1632 ± 12              | 2.1884 ± 0.0035            |
| 39  | 1734 ± 13              | 2.1865 ± 0.0037            |
| 40  | 1840 ± 15              | 2.1846 ± 0.0035            |
| 41  | 1949 ± 16              | 2.1830 ± 0.0032            |
| 42  | 2062 ± 18              | 2.1814 ± 0.0034            |
| 43  | 2178 ± 20              | 2.1801 ± 0.0034            |
| 44  | 2299 ± 23              | 2.1786 ± 0.0034            |
| 45  | 2422 ± 26              | 2.1773 ± 0.0035            |
| 46  | 2549 ± 28              | 2.1761 ± 0.0035            |
| 47  | 2694 ± 31              | 2.1753 ± 0.0039            |
| 48  | 2814 ± 34              | 2.1737 ± 0.0042            |
| 49  | 2952 ± 36              | 2.1726 ± 0.0045            |
| 50  | 3092 ± 38              |                            |

of descendents of a tree of size  $n$  is a direct estimate of  $(A_{n+1}/A_n)$ . A least-square fit of the form

$$A_{n+1}/A_n = \lambda(1 - \theta/n)$$

$$\log\langle I_{n,pl} \rangle = (2\nu_{pl} + 1) \log n + A$$

for  $n$  lying between  $n = 25$  and  $50$  gives the estimates

$$\lambda \approx 2.117 \pm 0.002$$

$$\theta \approx -1.28 \pm 0.02$$

$$2\nu_{pl} = 1.34 \pm 0.03.$$

These estimates are in fair agreement with the exact series estimates given earlier.

We note that the exponents for the square lattice spiral trees are quite different from those of spiral animals (loops allowed). For the latter problem, Bose and Ray (1987) have found that  $\theta \approx 0$ ,  $\nu \approx \frac{1}{2}$ . In contrast, in both the ordinary and directed variants, animals with or without loops are expected to be in the same universality class and have the same exponents (Gaunt *et al* 1982, Duarte 1985).

A scrutiny of the numerical data leads us to the following conjecture for the rooted spiral trees in dimensions  $d > 2$ :

$$\theta = (d - 4)\nu_{\perp} \quad \text{for } d > 2. \quad (\text{A})$$

If we use the classical values of  $\theta$  and  $\nu_{\perp}$  equal to the Cayley tree values  $\theta = \frac{3}{2}$ ,  $\nu_{\perp} = \frac{1}{4}$ , this equation implies that the upper critical dimension for the spiral trees problem is 10.

The case of  $d = 2$  is somewhat special. In this case  $\nu_{\perp}$  is not defined, and the only length scale in the problem is the in-plane extent of the spiral trees. It seems plausible to conjecture that, in this case, the above relation is modified to

$$\theta = (d - 4)\nu_{pl} \quad \text{for } d = 2. \quad (\text{B})$$

These relations (A) and (B) were suggested by the existence of similar relations for the undirected and directed animals:

$$\theta = (d - 2)\nu \quad \text{for undirected rooted trees}$$

$$\theta = (d - 1)\nu_{\perp} \quad \text{for directed rooted trees.}$$

These relations are the analogues of the conventional hyperscaling relations. For directed trees (as for directed animals) the factor  $(d - 1)$  is understood as being due to the dimensional reduction by one occurring when the disorder or cellular automaton condition is satisfied (see Rujan (1987) for a discussion and earlier references). For the undirected animals, the dimensional reduction by two is understood as being due to a hidden supersymmetry of the problem (Parisi and Sourlas 1981). For our problem, a similar explanation is not yet available. However, we suggest the following heuristic argument for the dimensional reduction by four.

A simple example where spiral motion leads to a dimensional reduction by two is the quantum mechanical motion of a charged particle in a uniform magnetic field. The degrees of freedom perpendicular to the field get quantised, and the resulting discrete spectrum does not contribute to the density of states of low-energy excitations. This leads to a reduction in the effective dimensionality of the system by two. In a field theoretic action for spiral animals, in addition to terms corresponding to independent charged-particle motion in an external magnetic field, there are terms corresponding to one particle giving rise to two particles (branching) and particle annihilation (branch ends). However, if these terms are treated as perturbations, the bare propagators appearing in the perturbation theory still show dimensional reduction by two. Thus this mechanism of dimensional reduction works term by term in a perturbation theory. The effective dimensionality is further decreased by two by the conventional Parisi-Sourlas (1981) mechanism.

Further studies of this problem are needed to elucidate fully the dimensional reduction in this problem and to understand its relationship with the conventional animal problem.

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